In sociology, someone’s power—the opportunity to influence others, possibly against their interest [4]—is regarded as being dependent on the power of those (s)he is socially related to. In other words, someone is powerful through her social contact’s power rubbing off on her. Phillip Bonacich [1, 2] turned this into a great idea by expressing it in a mathematical model, which substantially enhanced its predictive potential. Without his model, it would have been just an idea, but nobody would be able to tell for sure if it were great.

Let’s go into it now. A given social network component with \( m \) actors we can represent as an adjacency matrix \( R \) (see [3] for an introduction to social networks). When an actor \( b \) defers to actor \( a \), e.g. by asking for help or advice, \( a \) can influence \( b \), or can call upon \( b \)’s reciprocal support if the need arises; \( a \)’s power thus depends on \( a \)’s “incoming” ties, \( R_{ba} \). In the remainder, however, we assume for all \( a \) and \( b \), \( R_{ba} = R_{ab} \), even though Bonacich’s model also holds for asymmetric ties. We also assume that \( R_{ba} \geq 0 \), while the model can be generalized to negative ties as well.

The relational power (or status) \( c_a \) of actor \( a \) can be expressed as a summation of the power (or status) of \( a \)’s contacts (and if \( b \) is not a contact of \( a \) then \( R_{ba} = 0 \)), wherein each contact’s power is multiplied by the strength, or value, of the tie connecting \( a \) and \( b \), presuming that through stronger social ties more power rubs off. If the latter is not true, set all tie values larger than some threshold to one. A parameter \( \beta \) indicates the magnitude of the overall effect, and \( a \)’s power (or status) can then be written as

\[
c_a = \beta(c_1 R_{1a} + c_2 R_{2a} + \cdots + c_m R_{ma})
\]

More concisely, \( c_a = \sum_{g=1}^{g=m} (\beta c_g) R_{ga} \), and with an additional parameter \( \alpha \) that Bonacich uses to set the average power equal to one,

\[
c_a = \sum_{g} (\alpha + \beta c_g) R_{ga}
\]  \hspace{1cm} (1)

Notice that if \( \beta = 0 \) and \( \alpha = 1 \), the above expression reduces to \( a \)’s degree.

Rather than studying only \( a \)’s power, we want a vector \( c(\alpha, \beta) \) expressing the power centrality scores of all actors in one stroke, for which we need matrix notation. Let \( I \) be the identity matrix and \( v \) a column vector of ones. Assuming that \( (I - \beta R) \) is invertible (which is non-trivial), equation (1) in matrix notation becomes

\[
c(\alpha, \beta) = \alpha(I - \beta R)^{-1} R v
\]  \hspace{1cm} (2)
Notice that \((1 - \beta)^{-1}\) is the “radius” or expected path length of power rubbing off. To interpret the above equation, and to see how the effect of power rubbing off fades away at longer path distances, Bonacich claimed that if \(|\beta| < 1/\lambda_i\), and \(\lambda_i\) is the largest eigenvalue of \(\mathbf{R}\),

\[
\alpha(\mathbf{I} - \beta \mathbf{R})^{-1} \mathbf{Rv} = \alpha \sum_{k=0}^{\infty} \beta^k \mathbf{R}^{k+1} \mathbf{v} \tag{3}
\]

For the proof of this claim, Bonacich referred to a mathematics textbook, and went on with what he had to say. I (Jeroen Bruggeman) searched for this proof, didn’t find it in the textbook he mentioned (perhaps I didn’t look careful enough), and subsequently asked Vincent Traag if he could reconstruct the proof. The proof below is what he produced, and I translated his Dutch into English. Since neither of us has authorship of what follows (the spectral theorem exists for a long time and its application along the lines sketched below probably as well) this essay is author-less, with most of the credits for Bonacich himself.

We first focus on the partial summations \(S_n = \sum_{k=0}^{n} \beta^k \mathbf{R}^{k+1} \mathbf{v}\) (and \(n \in \mathbb{N}\)). We know that if the sequence \(S_n\) converges, the series has a limit equal to \(\lim S_n\), hence the summation in equation (3) is equal to \(\alpha \lim S_n\). We can write out this summation,

\[
S_n = \mathbf{Rv} + \beta \mathbf{R}^2 \mathbf{v} + \beta^2 \mathbf{R}^3 \mathbf{v} + \cdots + \beta^n \mathbf{R}^{n+1} \mathbf{v}
\]

and if we multiply by \(\beta \mathbf{R}\) we obtain

\[
\beta \mathbf{RS}_n = \beta \mathbf{R}^2 \mathbf{v} + \beta^2 \mathbf{R}^3 \mathbf{v} + \cdots + \beta^n \mathbf{R}^{n+1} \mathbf{v} + \beta^{n+1} \mathbf{R}^{n+2} \mathbf{v}
\]

If we subtract these two sequences, only two terms remain, \(\mathbf{Rv} + \beta^{n+1} \mathbf{R}^{n+2} \mathbf{v}\), and we can say that

\[
(\mathbf{I} - \beta \mathbf{R})S_n = \mathbf{Rv} + \beta^{n+1} \mathbf{R}^{n+2}
\]

or equivalently,

\[
S_n = (\mathbf{I} - \beta \mathbf{R})^{-1}(\mathbf{Rv} + \beta^{n+1} \mathbf{R}^{n+2})
\]

for which we have to prove that

\[
\lim_{n \to \infty} \beta^{n+1} \mathbf{R}^{n+2} = \mathbf{0} \tag{4}
\]

Because \(\mathbf{R}\) is symmetric, the spectral theorem has it that there exists an (orthogonal) \(\mathbf{S}\) and a diagonal matrix \(\mathbf{D}\) with the eigenvalues of \(\mathbf{R}\) on its diagonal such that

\[
\mathbf{R} = \mathbf{S}^{-1} \mathbf{DS}
\]

By now applying the spectral theorem to \(\mathbf{R}^2\) we get

\[
\mathbf{R}^2 = \mathbf{RR} = \mathbf{S}^{-1} \mathbf{DSS}^{-1} \mathbf{DS} = \mathbf{S}^{-1} \mathbf{D}^2 \mathbf{S}
\]
and by induction,

\[ R^n = S^{-1}D^nS \]

To prove conjecture (4), we will show that for \(|\beta| < 1/\lambda|,\)

\[
\lim_{n \to \infty} \beta^{n+1}D^{n+2} = 0
\] (5)

To go where we want to, we zoom in on \(D, \)

\[
\begin{pmatrix}
\beta^{n+1} \lambda_1^{n+2} & 0 & \cdots & 0 \\
0 & \beta^{n+1} \lambda_2^{n+2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta^{n+1} \lambda_m^{n+2}
\end{pmatrix}
\]

By introducing \(\delta = 1/\beta,\) hence \(|\delta| > \lambda_i,\) we obtain

\[
\lim \beta^{n+1} \lambda_j^{n+2} = \lim \frac{\lambda_j^{n+2}}{\delta^{n+1}} = \lim \lambda_j \left( \frac{\lambda_j}{\delta} \right)^{n+1}
\]

and because \(|\delta| > \lambda_i,\) and \(\lambda_i \geq \lambda_j\) for each \(j,\)

\[
\lim \lambda_j \left( \frac{\lambda_j}{\delta} \right)^{n+1} = 0 = \lim \beta^{n+1} \lambda_j^{n+2}
\]

Therefore

\[
\lim_{n \to \infty} \beta^{n+1}D^{n+2} = 0
\]

and consequentially

\[
\lim S_n = (I - \beta R)^{-1}Rv
\]

and therefore also

\[
\alpha \lim S_n = \alpha(I - \beta R)^{-1}Rv
\]

which is what we had to prove. The sociological interpretation is discussed in Bonacich’ papers.

References


